On universality between two-dimensional Ising-like systems

1987 J. Phys. A: Math. Gen. 20 3419

(http://iopscience.iop.org/0305-4470/20/11/043)

J. Phys. A: Math. Gen. **20** (1987) 3419-3424. Printed in the UK

On universality between two-dimensional Ising-like systems

Joan Adler

Department of Physics and Astronomy, Tel Aviv University, Ramat Aviv, Israel and Physics Department, Technion, Haifa, Israel

Received 15 April 1986, in final form 16 January 1987

Abstract. Questions concerning the universality between different two-dimensional Ising-like systems have recently been raised, with evidence for the existence of several different subclasses of Ising-like systems being presented. In this paper some new facts concerning similarity between border model and low-temperature spin-1 Ising model series analyses are noted. For the spin-1 case an alternative determination of the critical temperature determines unequivocally that this model is in the Ising universality class and it is suggested that a similar independent result could lead to a similar conclusion in the border model case. The possibility of an analytic correction to scaling in the spin-1 model is proposed.

The conventional wisdom that all two-dimensional Ising-like systems should have the same dominant critical exponents has recently been challenged by Baker and Johnson (1984, hereafter denoted as BJ) who propose that the critical exponent of the suceptibility, γ, has a range $1.89 < \gamma < 2.02$ for a model of the Ising 'universality' class that they call the 'border' model, whereas in the spin-$\frac{1}{2}$ Ising model $\gamma = 1.75$ exactly. The hard square lattice gas and spin $S = 1$ Ising model (Adler and Enting 1984, hereafter referred to as I), which are also presumed to be in the Ising class, also give $\gamma \sim 1.75$, although these do differ from the spin-$\frac{1}{2}$ case in that non-analytic corrections to scaling (estimated to have a correction exponent Δ_1, in the range $1 < \Delta_1 < 1.3$, spin 1, and $1.2 < \Delta_1 < 1.4$, hard square) are apparently present. In this paper we reconsider the analysis of the spin-1 low-temperature (LT) susceptibility series made in I since this analysis was partially based on the assumption that $\gamma = 1.75$, and show that for both this series and for the border model (γ, Δ_1) ranges from $(2.0, 0.9)$ to $(1.75, 1.2)$ as the critical temperature, T_c, varies. We then make an independent estimate of T_c for the spin-1 model from a new 21-term series for the high-temperature (HT) susceptibility on the square lattice and show that this new T_c estimate implies $\gamma = 1.75$. We also find indications of a possible analytic correction to scaling ($\Delta = 1$) in addition to a $\Delta_1 = 1.35$ term near this new T_c estimate. After completing these calculations we received a preprint from Barma and Fisher ((1984, 1985), hereafter denoted BF) who used partial differential approximants to study 21-term series for the Klauder, double Gaussian and ϕ^4 models which are also believed to fall within the Ising universality class. They also observed a $\gamma_{\rm eff} = 2.0$ for certain parameter choices but attributed it to crossover phenomena and favour Ising-like criticality with a correction exponent of $\Delta_1 = 1.35 \pm 0.25$. They do not appear to report any indication of an analytic correction for any of the models that they studied. Blöte and Nightingale (1985) used the transfer matrix formalism to confirm that the spin-1 Ising model fits accurately in the Ising universality class.

The critical behaviour of the susceptibility for all models in the Ising class is expected to take the form

$$\chi(T>T_c) \sim (K-K_c)^{-\gamma}(1+a_{H1}(K-K_c)+a_{\Delta_{H1}}(K-K_c)^{\Delta_1}+\ldots)$$
$$\chi(T<T_c) \sim (u_c-u)^{-\gamma}(1+a_{L1}(u_c-u)+a_{\Delta_{L1}}(u_c-u)^{\Delta_1}+\ldots)$$

where $K=3J/2kT$ and $u=\exp(-J/kT)$ are the HT and LT variables. The spin-$\frac{1}{2}$ model has no term with $\Delta_1<1$ and a_{H1} and a_{L1} are $\neq 0$. Expansions for $\chi(T<T_c)$ for the spin-1 model can be found in I and 21 terms of $\chi(T>T_c)$ were derived by Nickel (1986). We note that while the LT series appear at first sight to be longer, they converge more slowly and thus the HT series should give a more reliable determination of T_c. It is by now well known that the non-analytic correction terms can influence the value of γ deduced from an analysis of the above series so we have studied both the spin-1 LT and HT series as well as the border model series of BJ with a method that gives graphs of γ as a function of Δ_1 for different T_c values. Details of the method of analysis are described in I so we move directly to the results.

In figure 1 we display the (γ, Δ) plane of the border model on the triangular lattice for three choices of K_c, namely the final K_c choice of BJ (1(*a*), $K_c=0.2134$), the central value given by the usual Padé analysis (1(*b*), $K_c=0.2130$) and a third value which is consistent with $\gamma=1.75$ (1(*c*), $K_c=0.2118$). Figure 2 illustrates the (γ, Δ) plane for similarly chosen K_c values for the square lattice. We have $K_c=0.3300$ (2(*a*)), $K_c=0.3290$ (2(*b*)) and $K_c=0.3275$ (2(*c*)). In both cases we observe that the value of (δ, Δ)

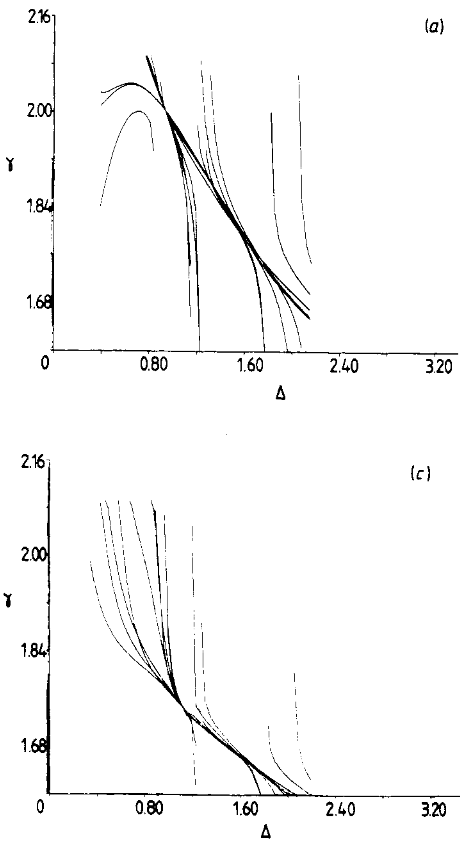
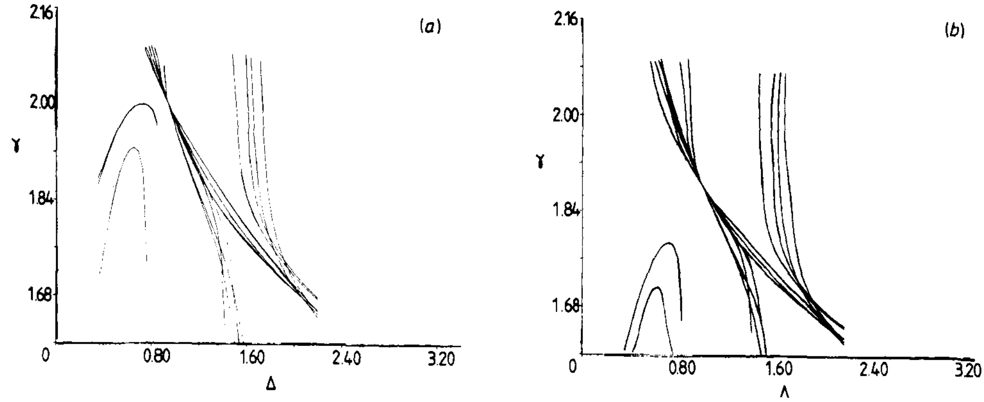


Figure 1. Graphs of γ against Δ for the HT border model series on the triangular lattice for (*a*) $K_c=0.2134$, (*b*) $K_c=0.2130$, (*c*) $K_c=0.2118$.



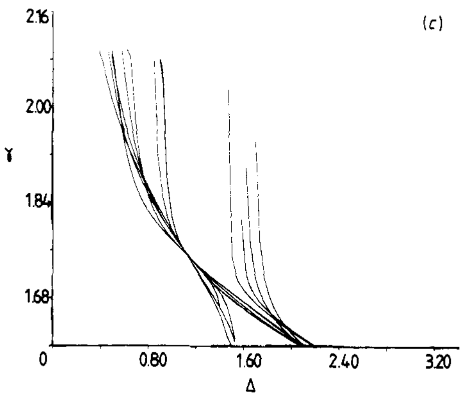


Figure 2. Graphs of γ against Δ for the HT border model series on the square lattice for (*a*) $K_c = 0.3300$, (*b*) $K_c = 0.3290$, (*c*) $K_c = 0.3275$.

at the K_c choice of BJ is consistent with the exponent results quoted by BJ and the exponents that can be read off the (*b*) plots are consistent with the Padé results. The (*c*) plots are consistent with a Δ_1 estimate of ~ 1.1. It would be presumptuous to claim that the convergence in the (*c*) plot is better than in the (*a*) plots, but it is certainly no worse.

Let us now turn to the LT spin-1 subsceptibility. We display the (γ, Δ) plane for $u_c = 0.555$ in figure 3 and observe that we would select the values $\gamma \sim 1.96$ and $\Delta \sim 0.8$ which are consistent with the results from figures 1(*a*) and 2(*a*). This figure can be

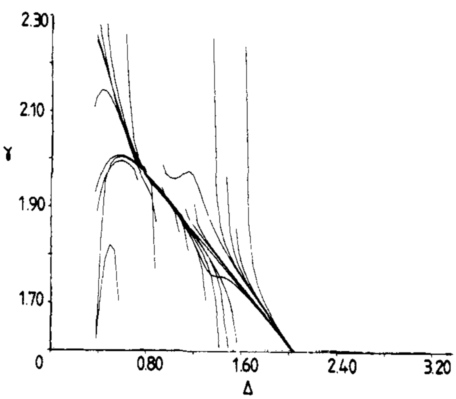


Figure 3. Graph of γ against Δ for the LT spin-1 Ising model series on the square lattice for $u_c = 0.555$.

compared with figure 1 of I which gives $\gamma = 1.75$, $\Delta_1 \sim 1.1$ at $u_c = 0.554\,06$, consistent with figures 1(*c*) and 2(*c*). Thus we may conclude that the spin LT series behaves similarly to the border model series and in fact if universality is in question then the assumption that $\gamma = 1.75$ for the spin-1 model and the resulting analysis made in I must be questioned.

Let us now consider how we can differentiate between the two sets of values, namely $\gamma > 1.8$, $\Delta_1 < 1$ and $\gamma = 1.75$, $\Delta_1 > 1$. Since K_c increases (u_c decreases) as γ increases, one way to decide would be to invoke an independent evaluation of K_c for one of the models, and this can be achieved for the spin-1 model by investigating the HT susceptibility. (We note that the LT magnetisation was consistent with $u_c = 0.554\,06$.) We begin our discussion by noting that in the HT case there is a fairly narrow range of K_c values for which we do see quite tight convergence regions. Three representative K_c values from this range, $0.3936 < K_c < 0.3937$, are discussed in detail below. In figure 4(*c*) we show the (γ, Δ) plane of the HT susceptibility for $K_c = 0.393\,65$, and observe intersection regions near $\Delta \sim 1$, (analytic (?), γ being just below 1.75 here) and we have $\gamma = 1.75$ at $\Delta \sim 1.35$. There is a third intersection region near $\Delta \sim 1.7$. For $K_c = 0.393\,675$ (figure 4(*b*)) we observe intersections at $\Delta = 1.2$ (γ just above 1.75) and $\Delta = 1.6$, and for $K_c = 0.393\,625$ (figure 4(*d*)) we have intersections at $\Delta \sim 0.9$ and $\Delta \sim 1.3$ ($\gamma < 1.75$ in both cases). If we look over a wider range of K_c values we see two broad trends: as K_c increases from $K_c \sim 0.393\,675$ there are two intersection regions (denoted by A and B on the figures) at Δ varying from 1.2 to 1.4 and $\Delta \sim 1.6$, 1.7, respectively and we show the (γ, Δ) plane for $K_c = 0.394$ in figure 4(*a*). As K_c decreases from $K_c \sim 0.393\,6625$ we observe three regions, denoted by C, D and E at $\Delta \leqslant 1$, $\Delta \approx 1.3$, 1.4 and $\Delta \geqslant 1.6$, respectively, and we illustrate the case of $K_c = 0.3933$ in figure 4(*e*). Below $K_c = 0.393\,300$ convergence is extremely poor. If we identify the A region of figures 4(*a*) and (*b*) with the D region of 4(*c*), (*d*) and (*e*), and the B region with the E region, we see that there is one intersection region that is quite stationary at $\Delta \sim 1.35$ and an intersection region C that shifts to increasing Δ values as K_c increases. Within the region of best convergence ($0.3936 < K_c < 0.3937$) this region joins with the A/D region for $0.393\,6625 < K_c < 0.393\,675$, passing through $\Delta = 1.0$ at $K_c = 0.393\,65$.

We conclude with a discussion of two points raised by the above analysis. The first concerns the K_c estimates. The tightest convergence in the HT spin-1 series is consistent with $\gamma = 1.75$ and $0.3936 < K_c < 0.3937$. This K_c range corresponds to $0.5540 < u_c < 0.5541$, totally consistent with the results of I for u_c that were based on the assumption $\gamma = 1.75$. Thus we see that in the spin-1 case the correct u_c gives $\gamma = 1.75$.

We note that our central u_c estimate corresponds to $K_c = 0.3905$ in the K_c units of Blöte and Nightingale and compares most favourably with their final estimate of $K_c = 0.590\,463$. If we consider the possible γ values from our range of K_c estimates we have $\gamma = 1.75 \pm 0.005$. We speculate that if an independent evaluation of K_c were available for the border model this would confirm the values consistent with $\gamma = 1.75$, namely $K_c = 0.2118$ (triangular lattice) and $K_c = 0.327\,25$ (square lattice). We note that the square lattice estimate falls within the Padé approximant range of BJ whereas the triangular lattice estimate does not.

The second point raised by this analysis concerns the identification of the different intersection regions A, C and D, observed in the analysis of the HT spin-1 susceptibility series, with each other and with the single intersection region observed in the other series. We may speculate that the C region corresponds to an analytic correction that apparently has a small amplitude and thus is not visible in the LT spin-1 or border model series. Alternatively the single region observed in these series may correspond

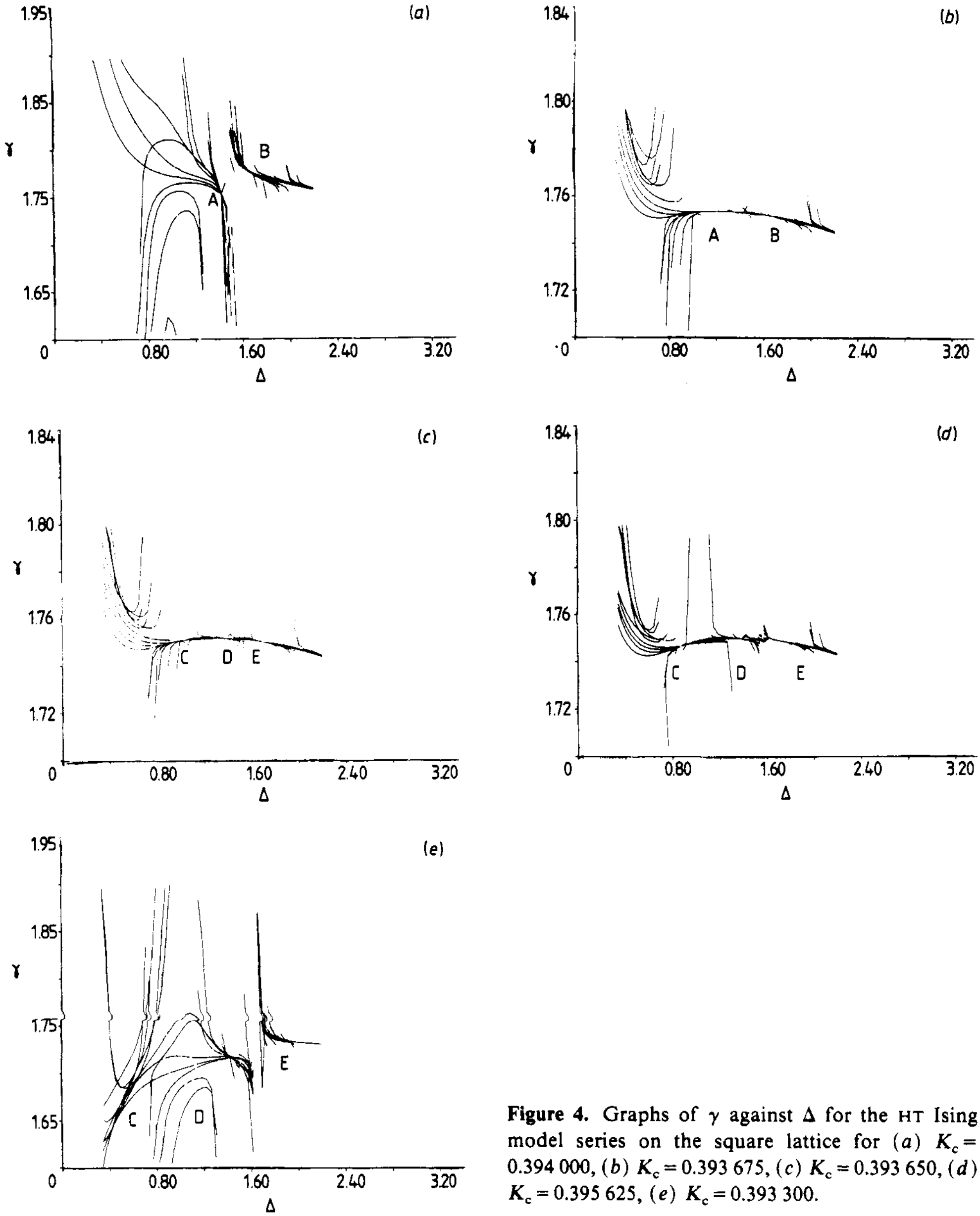


Figure 4. Graphs of γ against Δ for the HT Ising model series on the square lattice for (a) $K_c = 0.394\,000$, (b) $K_c = 0.393\,675$, (c) $K_c = 0.393\,650$, (d) $K_c = 0.395\,625$, (e) $K_c = 0.393\,300$.

to the C region and the D region may not be observable there. Perhaps the exponent $\Delta_1 \sim 1.1$ is an effective exponent that averages the C and D regions of the HT spin-1 series. This question could be resolved either by a study of the series for $s > 1$ or by scaling arguments for or against an analytic term in the spin-1 model.

Acknowledgments

The author acknowledges the support of the Centre for Absorption in Science of the Government of Israel. We thank M Barma and G Baker for forwarding preprints prior to publication. We thank B Nickel for kindly providing the spin-1 high-temperature series.

References

Adler J and Enting I G 1984 *J. Phys. A: Math. Gen.* **17** 2233
Baker G A and Johnson J D 1984 *J. Phys. A: Math. Gen.* **17** L275
Barma M and Fisher M E 1984 *Phys. Rev. Lett.* **53** 1935
—— 1985 *Phys. Rev.* B **31** 5954
Blöte H W J and Nightingale M P 1985 *Physica* **134A** 274
Nickel B 1986 Private communication